Exercise 6

Find the general solution for each of the following first order ODEs:

$$xu' - u = x^2 \sin x, \; x > 0$$

Solution

First rewrite the differential equation so that the coefficient of u' is 1.

$$u' - \frac{1}{x}u = x \sin x$$

This is an inhomogeneous first order linear ODE, so we can multiply both sides by the integrating factor,

$$I(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = x^{-1},$$

to solve it. The equation becomes

$$x^{-1}u' - x^{-2}u = \sin x.$$

Observe that the left side can be written as $(x^{-1}u)'$ by the product rule.

$$\frac{d}{dx}(x^{-1}u) = \sin x$$

Now integrate both sides with respect to x.

$$x^{-1}u = -\cos x + C$$

Therefore,

$$u(x) = x(C - \cos x), \; x > 0.$$